Local conversion of GHZ states to approximate W states

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Abstract

Genuine 3-qubit entanglement comes in two different inconvertible types represented by the GHZ state and the W state. We describe a specific method based on local positive operator valued measures (POVMs) and classical communication that can convert the ideal N-qubit GHZ state to a state arbitrarily close to the ideal N-qubit W state. We then experimentally implement this scheme in the 3-qubit case and characterize the input and the final state using 3-photon quantum state tomography.

Entanglement is at the heart of quantum mysteries and the power of quantum information. While 2-qubit entanglement is fairly well understood with good entanglement measures [1], understanding multi-qubit entanglement remains a considerable challenge. With 3-qubits it is no longer enough simply to ask *if* the qubits are entangled or not; one must also ask *how* the qubits are entangled. The two most common examples of 3-qubit entangled states are the $|\text{GHZ}\rangle = 1/\sqrt{2}\,(|HHH\rangle + |VVV\rangle)$ [2] and $|\text{W}\rangle = 1/\sqrt{3}\,(|HVV\rangle + |VHV\rangle + |VVH\rangle)$ [3, 4], where $|H\rangle$ and $|V\rangle$ represent horizontally- and vertically- polarized photon states. An important characteristic of 3-particle GHZ states is that loss of any one of the qubits leaves the other two in a mixed state with only classical correlations. It is well-known that if two qubits are maximally entangled, then neither can be entangled to a third; the W state is the 3-qubit state in which each pair of qubits have the same and maximum amount of bipartite entanglement. This feature makes the entanglement of the W state maximally symmetrically robust against loss of any single qubit. It has been shown that the states $|\text{GHZ}\rangle$ and $|\text{W}\rangle$ represent two distinct classes of 3-qubit entanglement that cannot be interconverted under any local operations and classical communication (LOCC)[4, 5]. Experimental realizations of GHZ states and more recently W states have been performed in optical and trapped ion experiments [6, 7, 8, 9, 10].

While conversion of a GHZ state to an exact W state is not possible via LOCC, a specific scheme based on partial quantum measurement (positive operator valued measures or POVMs) and classical communication can, however, convert a GHZ state to an *arbitrarily good approximation* to a W state with a tradeoff between the fidelity of the final state and the probability of success. In any real experiment, where there is inevitable noise in state production and measurement, arbitrarily good approximations are indistinguishable from directly-prepared W states. In the present work, we first discuss a POVM scheme, inspired by the procrustean method [11], for converting the 3-qubit state $|\text{GHZ}\rangle$ into an approximate $|\text{W}\rangle$. Then we generalize the scheme to convert between N-qubit analogues of the GHZ and W states. Finally, we experimentally apply the POVM scheme to a 3-photon GHZ state and characterize the change from the input to the output using quantum state tomography [12].

In the diagonal basis, where $|D\rangle = 1/\sqrt{2}\,(|H\rangle + |V\rangle)$ and $|A\rangle = 1/\sqrt{2}\,(|H\rangle - |V\rangle)$, we can rewrite $|\text{GHZ}\rangle = 1/2\,(|DDD\rangle + |DAA\rangle + |ADA\rangle + |AAD\rangle)$. It is apparent in this basis that $|\text{GHZ}\rangle$ is a superposition of an unwanted term, $|DDD\rangle$, and a W-state. We

define a local POVM with elements $\varepsilon_1 = |A\rangle\langle A| + a^2 |D\rangle\langle D|$ and $\varepsilon_2 = (1-a^2)|D\rangle\langle D|$, where a is a real number between 0 (perfect measurement) and 1 (no measurement). This POVM is applied to each photon in the state and if all of the parties find element ε_1, then the new state is $|\psi\rangle = \mathcal{N}\left[a^3/2\,|DDD\rangle + a\sqrt{3}/2\,|\mathrm{W}'\rangle\right]$,with the normalization constant, $\mathcal{N} = 2/\sqrt{a^6+3a^2}$. The state $|\mathrm{W}'\rangle = 1/\sqrt{3}\,(|DAA\rangle + |ADA\rangle + |AAD\rangle)$ is simply related to $|\mathrm{W}\rangle$ by three single-qubit rotations. It is clear from this state, that the unwanted term $|DDD\rangle$ is reduced relative to the term $|\mathrm{W}'\rangle$. It is also clear that one achieves a pure $|\mathrm{W}'\rangle$ only in the limit as $a \to 0$ where the probability of success also goes to zero. Nevertheless, the fidelity of this state with the desired $|\mathrm{W}'\rangle$ is $\mathcal{F}_{\mathrm{W}'} = |\langle\psi|\mathrm{W}'\rangle|^2 = 3/(a^4+3)$, which rapidly rises from 3/4 to 1 as a decreases, i.e., as the strength of the measurement increases. Conversely, the fidelity with the GHZ state $\mathcal{F}_{\mathrm{GHZ}} = |\langle\psi|\mathrm{GHZ}\rangle|^2 = (a^4+6a^2+9)/(4a^4+12)$ drops from 1 to 3/4 as a decreases.

Our method can be generalized to convert an N-qubit GHZ state, $|N+\rangle$, where $|N\pm\rangle = 1/\sqrt{2}\left(|H\rangle^{\otimes N} \pm |V\rangle^{\otimes N}\right)$, into an arbitrarily good approximation to the N-qubit W state $|\mathrm{W}'_N\rangle = 1/\sqrt{N}\,(|DA...A\rangle + |AD...A\rangle + ... + |AA...D\rangle)$ [13]. We use the fact that the GHZ states $|N\pm\rangle$ satisfy the following relation:

$$|N\pm\rangle = \frac{1}{\sqrt{2}}\left[|(N\text{-}M)+\rangle\,|M\pm\rangle + |(N\text{-}M)-\rangle\,|M\mp\rangle\right], \tag{1}$$

where $M < N$. Notice that this factorization preserves the evenness or oddness in the number of negative signs. Through repeated application of these two rules, one can factor $|N+\rangle$ in terms of only single-qubit states $|D\rangle \equiv |1+\rangle$ and $|A\rangle \equiv |1-\rangle$. This reexpresses the GHZ state as an equally weighted superposition of *all* 2^{N-1} terms with an *even number* of $|A\rangle$s. When N is odd, the GHZ state can be directly rewritten as,

$$|N+\rangle = \frac{1}{\sqrt{2^{N-1}}}\left[\sqrt{N}\,|\mathrm{W}'_N\rangle + \sqrt{2^{N-1}-N}\,|\phi\rangle\right], \tag{2}$$

where the state $|\phi\rangle$ is a superposition of all those terms containing an odd number and at least 3 $|D\rangle$s. When N is even, application of a local transformation $|D\rangle \to |A\rangle$, $|A\rangle \to |D\rangle$ to any qubit allows the GHZ state to be written in the form of Eq. 2. Applying the same local POVM as in the 3-qubit case on each of the N qubits, and given that each POVM returns element ε_1 the unwanted amplitudes by *at least* a factor of a^3 while reducing the desired amplitude by only a single factor of a. In general, the fidelity of the resultant state

by this prescription with $|\mathrm{W}'_N\rangle$ is given by

$$\mathcal{F}^N_{\mathrm{W}'} = \frac{2a^2 N}{\left(1+a^2\right)^N - \left(1-a^2\right)^N}$$

regardless of whether N is even or odd.

The details on our experimental method for creating 3-photon GHZ states can be found in [14]. Ultraviolet laser pulses from a frequency-doubled Ti:Sapphire laser make two passes through a type-II phase-matched β-barium borate (BBO) crystal aligned, with walk-off compensation to produce 2-photon pairs each in the Bell state $\left|\phi^+\right\rangle$ [15]. These 2 independent photon pairs can be further entangled when combined at the polarizing beamsplitter (PBS1) and the four photons take four separate outputs A and B. Recall that a PBS works by reflecting horizontally-polarized light $|H\rangle$ and transmitting vertically-polarized light $|V\rangle$. Thus, two photons that were incident from different sides can only pass into different output modes when their polarizations were both $|H\rangle$ or both $|V\rangle$. In this sense, the PBS acts as a quantum parity check [16]. Given that the parity check succeeds on the two photons from the independent pairs, our state is transformed from the product state $\left|\phi^+\right\rangle_{12}\left|\phi^+\right\rangle_{34}$ to the 4-photon GHZ state $|4+\rangle = 1/\sqrt{2}\left(|HHHH\rangle_{AB14} + |VVVV\rangle_{AB14}\right)$ [8]. We project photon 4 onto the state $|D\rangle$ and when this projection succeeds leaves $|\mathrm{GHZ}\rangle = 1/\sqrt{2}\left(|HHH\rangle_{AB1} + |VVV\rangle_{AB1}\right)$.

A tomographically complete set of measurements for a 3-photon polarization state requires 64 polarization measurements. We use the 64 combinations of the single-photon projections $|H\rangle$, $|V\rangle$, $|D\rangle$, and $|R\rangle$ on each of the 3 photons. These projections are implemented using a quarter-wave plate and polarizer for each of photons A and B, and a half- or quarter-wave plate and PBS2 for photon 1. Successful projections are signalled by four-photon coincidence measurements, 3 photons for the state and 1 trigger photon, using single-photon counting APDs and coincidence logic. The most-likely physical density matrix for our 3-qubits is extracted using maximum-likelihood reconstruction [17, 18].

We begin with the GHZ state that was characterized previously via 3-photon quantum state tomography [14]. We rewrite the density matrix in the $|D/A\rangle$ basis; this gives the density matrix shown in Fig. 2b (real part) and 2c (imaginary part). A comparison to the ideal GHZ written in the same basis is shown in Fig. 2a (real part only, the imaginary part is all zero). The colour plots display the absolute value of each element and show that the two matrices have the same structure.

Our POVM was implemented using three partial polarizers. Instead of orienting the polarizers in the $|D/A\rangle$ basis, we rotated the polarization of each photon by 45° using half-wave plates. These rotations were accomplished by using the existing half-wave plate in mode 1, and by adding two additional half-wave plates in modes A and B. These extra rotations allowed the partial polarizers to operate in the $|H/V\rangle$ basis, and therefore our POVM also operates in the $|H/V\rangle$ basis. Each polarizer comprised of two uncoated glass microscope slides such that the angle of incidence for the input light was at 56°, near Brewster's angle (Figure 1). The configuration of the plates are such that the beam experienced minimal additional transverse shift and maintained high coupling efficiency into single-mode fibres. Such partial polarizers have been used to study hidden nonlocality and entanglement concentration of maximally-entangled mixed states [19]. We placed each such element so that vertically-polarized light was P-polarized and horizontally-polarized light was S-polarized; we measured 88% transmission for the vertically-polarized light and 33% for the horizontally-polarized light. The transmission of the vertical light is thus only 38% of that for the horizontal and we can can describe the experiment using the POVM elements ε_1 and ε_2 with $a^2 \approx 38\%$. With this attenuation value, and beginning with the ideal GHZ state, the fidelity of the state with the ideal W state given 3 POVM outcomes ε_1 is expected to increase from 75% to 95%.

We used the same 64 tomographic measurement settings for the W state as for the GHZ state. Data for each setting was accumulated for 1800 seconds and yielded a maximum of 120 four-fold coincidence counts (for the $|VVV\rangle$ projection). To account for laser power drift, which was small but not insignificant, we divided the four-folds by the square of the singles at the trigger detector. Background four-folds from a two-fold coincidence count and an uncorrelated accidental were estimated for each measurement setting and subtracted from the measured coincidences. Using the maximum likelihood reconstruction, our most likely density matrix is shown in Fig. 3b (real part) and 3c (imaginary part). The ideal W state density matrix consists of only 9 real elements – 3 diagonals of 1/3 height corresponding to $|HVV\rangle$, $|VHV\rangle$, and $|VVH\rangle$ and 6 maximal positive coherences between them. It is clear from the data that the dominant elements in the density matrix are those same 9 elements. In Fig. 3a., we show the effect of the POVM with our experimentally measured attenuation on the ideal GHZ state. The diagonal elements are attenuated much more strongly, by a^2, while the coherences remain maximal and are thus reduced only by a. Note that one specific

unwanted term contained in the diagonal element for $|VVV\rangle$ is much more significant after the application of the POVM; this noise contribution, in the ideal case, is untouched by the POVM, and in the real case, least reduced.

We characterize the changes in our states using fidelity. The fidelity of a density matrix, ρ, with a pure quantum state, $|\psi\rangle$, is given by $\mathcal{F} = \langle\psi|\,\rho\,|\psi\rangle$. We calculate this fidelity with a general GHZ (W) state, $|\mathrm{GHZ}_G\rangle$ ($|\mathrm{W}_G\rangle$), which is related to $|\mathrm{GHZ}\rangle$ ($|\mathrm{W}\rangle$) by 3 local unitary rotations. The initial state has a fidelity of $\mathcal{F}_{\mathrm{GHZ}_G} = (79.4 \pm 1.6)\,\%$ and $\mathcal{F}_{\mathrm{W}_G} = (60.5 \pm 1.9)\,\%$ as compared with the ideal 100% and 75%. After successful application of three local POVMs, $\mathcal{F}_{\mathrm{GHZ}_G} = (59.8 \pm 2.5)\,\%$ and $\mathcal{F}_{\mathrm{W}_G} = (68.4 \pm 2.4)\,\%$. Uncertainties in quantities extracted from these density matrices were calculated using a Monte Carlo routine and assumed Poissonian errors. A theoretical calculation based on our measured initial state and measured a has yields a final state with $\mathcal{F}_{\mathrm{W}_G} = 75\%$. Thus much of the difference with the expected fidelity in the ideal case is a result of the quality of the initial state. Nevertheless, the overlap with a W state has been significantly improved while the overlap with a GHZ state has been strongly reduced.

We have described a method for converting N-qubit GHZ states to arbitrarily good approximations to N-qubit W states based on generalized quantum measurements (POVMs). We have implemented this scheme for the 3-qubit case and characterized the input and output states using multiphoton quantum state tomography. We have quantitatively shown that the transformation induced by the partial polarizers results in a decrease in the overlap of the state with a GHZ state while increasing the overlap with the desired W state. Multiparticle entanglement is essential to the success of quantum information processing. The theory and experimental work presented here extends our abilities to manipulate and understand the relationship between different types of complex entangled states.

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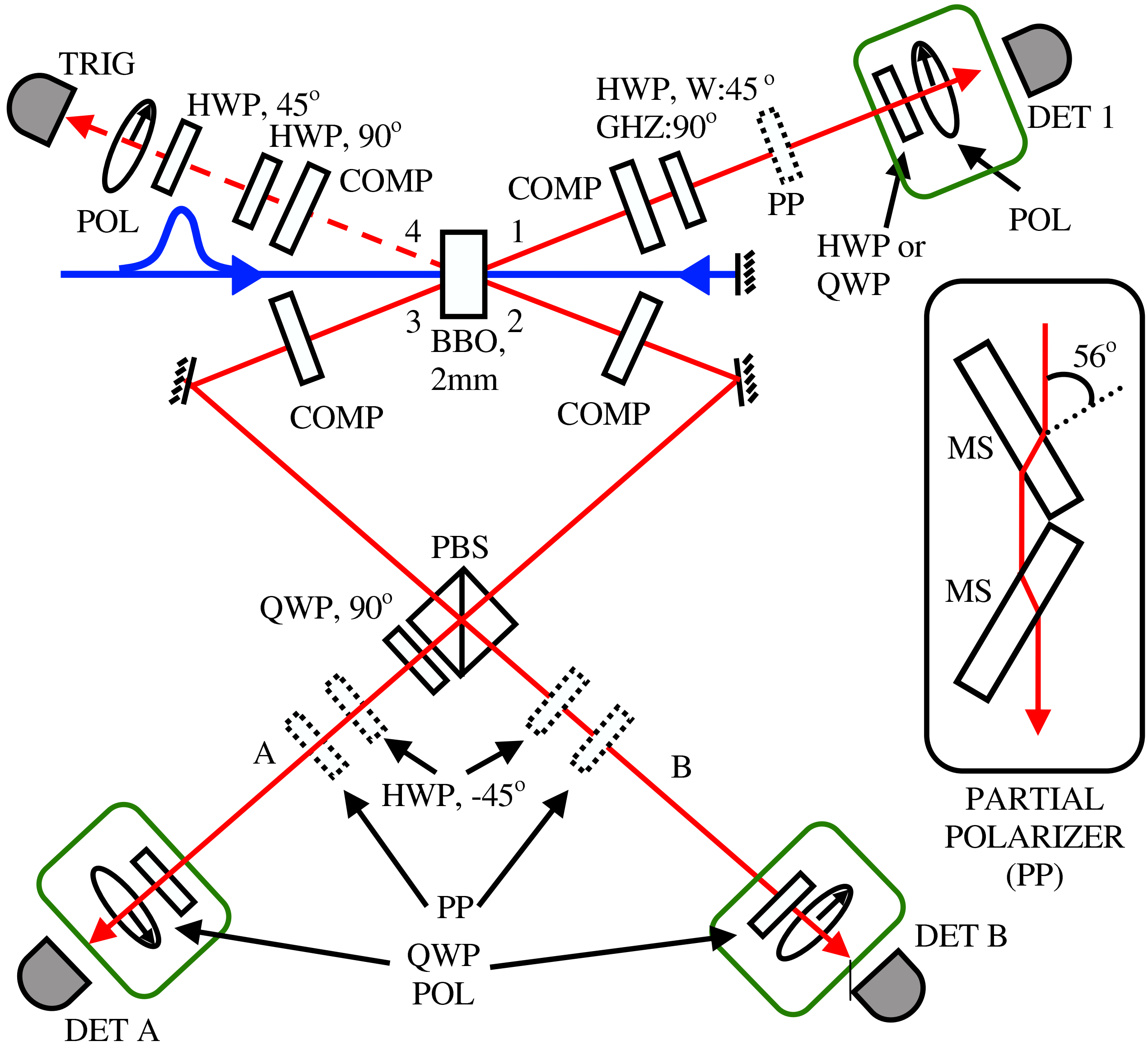
Figure 1. Experimental setup for production of GHZ state and its conversion to an approximate W state. A double-pass pulsed parametric down-conversion crystal (BBO) is used to create two pairs of polarization-entangled photons both in the state $\left|\phi^{+}\right\rangle$. Extra crystals (COMP) compensate for walkoff effects. A polarizing beamsplitter (PBS) performs a parity check on two of the photons, one from each pair; given that the parity check succeeds, signalled by the two photons taking different output modes, the emerging photons are in a four photon GHZ state. Projection of photon 4 onto the state $|D\rangle$ leaves the remaining 3 photons in the desired 3-photon GHZ state. Each photon in the GHZ state was rotated locally – photons A and B were rotated using additional half-wave plates (HWP) and photon 1 was rotated using the existing half-wave plate, rotating the polarization only $45°$ instead of $90°$. Our 3-qubit local POVM is performed using 3 partial polarizers (PP). Each partial polarizer consists of two microscope slides (MS) mounted such that the light is incident at $56°$, Brewster's angle for $n = 1.5$. This configuration had a measured transmission of 88% for p-polarization and 33% for s-polarization. Each POVM was oriented to reduce the horizontal component of the light relative to the vertical component. The three-photon polarization measurements for tomography were taken using quarter-wave plates (QWP) and rotatable polarizers (POL) for photons A and B, and a half- or quarter-wave plate and a fixed polarizer (actually a second PBS) for photon A before photon-counting detectors (DET and TRIG). A fixed QWP is mode A was used to compensate birefringence in the PBS.

Figure 2. Density matrix of an ideal and experimentally measured 3-photon GHZ state. The reconstructed density matrix of an a) ideal GHZ state (real part only) and our measured GHZ state, real part b) & imaginary part c), from reference [14]. The state is displayed in the D/A-basis, where $D = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and $A = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$ are defined as in the text. The false colour plots display the absolute value of each component of the matrices and are meant to show the structure of the matrix - namely that our GHZ state is characterized in this basis by 4 diagonal elements of equal height with maximal positive coherences. The experimentally measured density matrix has a fidelity of 77% with the ideal

GHZ state and 79% with any state related to the ideal GHZ state via local unitary transformations.

Figure 3. Density matrix of approximate W output state after the POVM procedure. The reconstructed density matrix of our output state, b) real part & c) imaginary part, after the three local POVM operations. The application of the POVMs have suppressed several of the matrix components such that the final state contains only 9 major elements. These are the same 9 elements for the ideal W state. The operation has increased the fidelity of our state with a W state from 61% to 68% while at the same time reducing the fidelity of our state with a GHZ state from 79% to 60%. For comparison, we show a) the action of the POVM operation on the ideal GHZ state. Although this state still contains large coherences with the HHH component, this state has 95% fidelity with the W state.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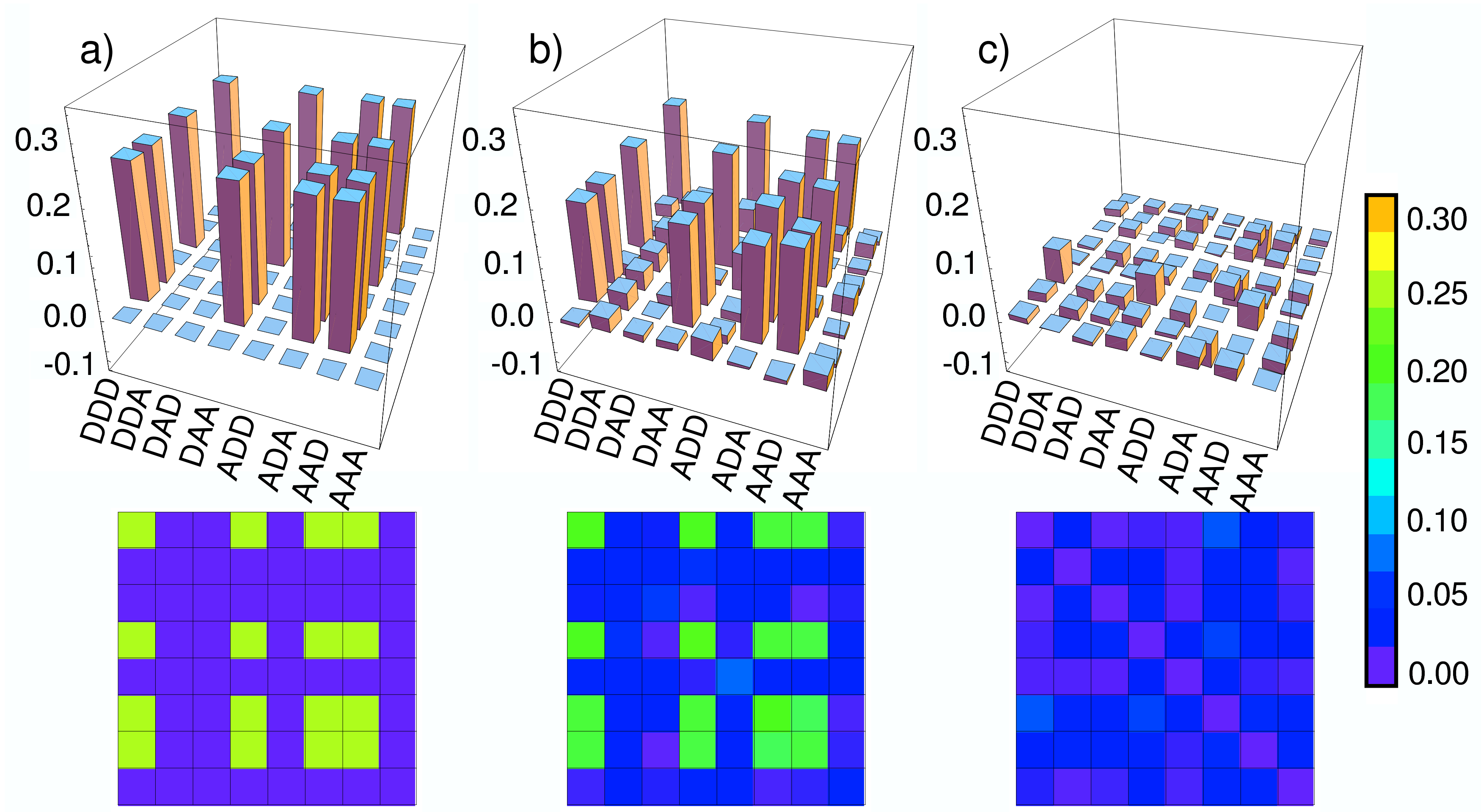
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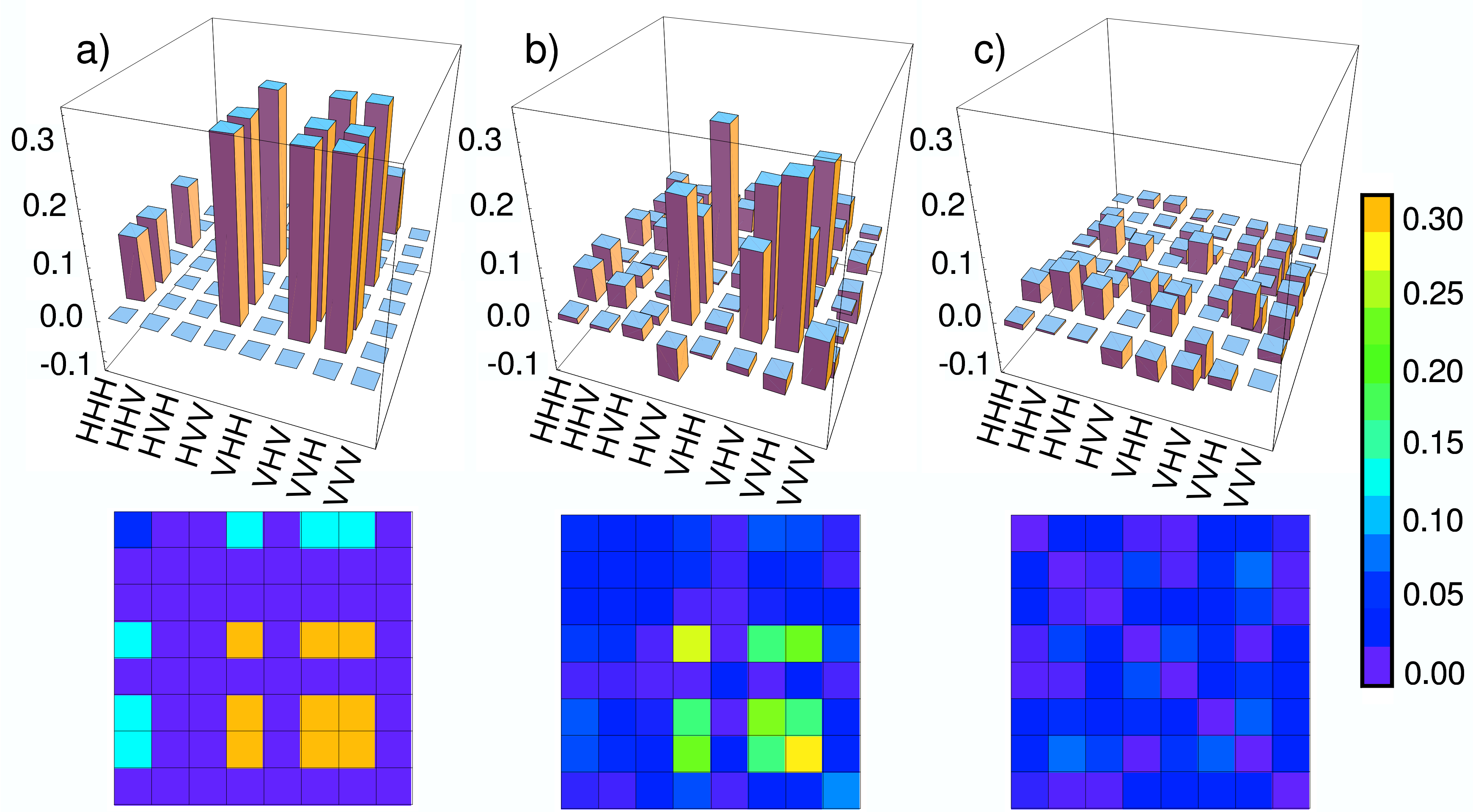

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